

Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

Outline for Day 13 - 14

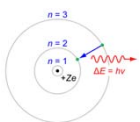
Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

Models of the Atom

- Thomson – “Plum Pudding”
 - Why? Known that negative charges can be removed from atom.
 - Problem: Doesn’t match scattering experiments
- Rutherford – Solar System
 - Why? Scattering showed a small, hard core.
 - Problem: electrons should spiral into nucleus in $\sim 10^{-11}$ sec.
- Bohr – quantized energy levels
 - Why? Explains spectral lines.
 - Problem: No reason for fixed energy levels



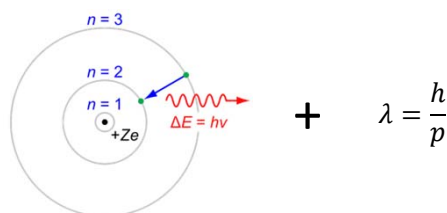
Successes of Bohr Model

- 'Explains' source of Balmer formula and predicts empirical constant R (Rydberg constant) from fundamental constants
- Explains why R is different for different single electron atoms (called *hydrogen-like ions*).
- Predicts approximate size of hydrogen atom
- Explains (sort of) why atoms emit discrete spectral lines
- Explains (sort of) why electron doesn't spiral into nucleus

Shortcomings of the Bohr model:

- Why is angular momentum quantized yet Newton's laws still work?
- Why don't electrons radiate when they are in fixed orbitals yet Coulomb's law still works?
- No way to know *a priori* which rules to keep and which to throw out...
- Can't explain shapes of molecular orbits and how bonds work
- Can't explain doublet spectral lines

Bohr Atom + de Broglie electron waves

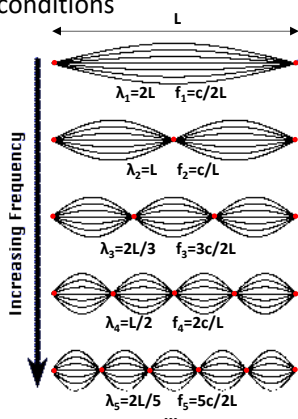


Waves with boundary conditions

- Boundary conditions mean that waves only have discrete frequencies.
- e.g., guitar strings

$$\lambda_n = 2L/n \quad f_n = nc/2L$$

• = node = fixed point that doesn't move

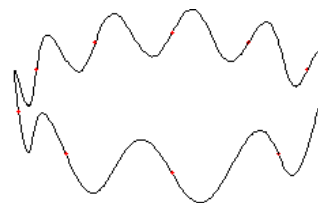


PHET: normal modes

Standing Waves on a Ring

Just like standing wave on a string,
but now the two ends of the string are joined.

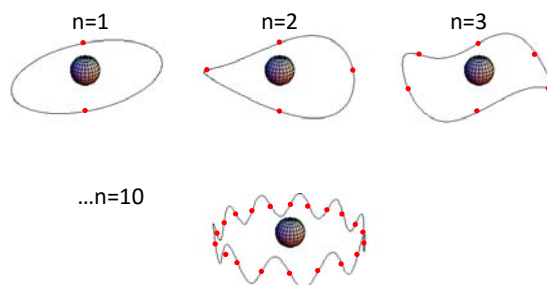
$$2\pi r = n\lambda \quad n = 1, 2, 3, \dots$$



Derive angular momentum of an electron wave

deBroglie electron waves in an atom

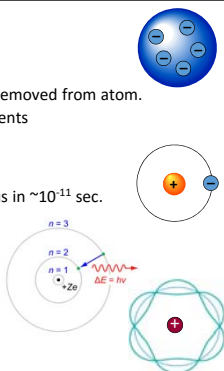
$$\lambda = \frac{h}{mv}$$



• = node = fixed point that doesn't move

Models of the Atom

- Thomson – “Plum Pudding”
 - Why? Known that negative charges can be removed from atom.
 - Problem: Doesn't match scattering experiments
- Rutherford – Solar System
 - Why? Scattering showed a small, hard core.
 - Problem: electrons should spiral into nucleus in $\sim 10^{-11}$ sec.
- Bohr – quantized energy levels
 - Why? Explains spectral lines.
 - Problem: No reason for fixed energy levels
- deBroglie – electron standing waves
 - Why? Explains quantized energy levels
 - Problem: still only works for Hydrogen.



Outline for Day 13 - 14

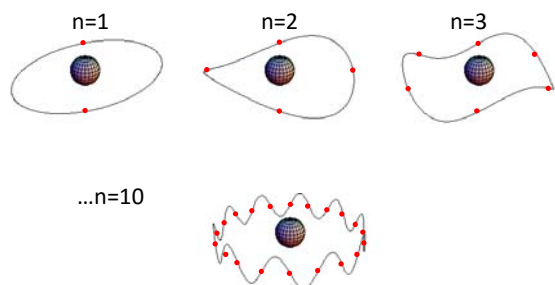
Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

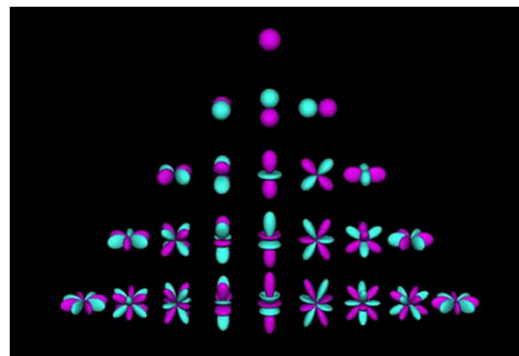
- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

deBroglie electron waves in an atom

$$\lambda = \frac{h}{mv}$$



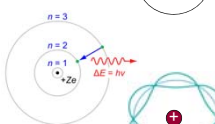
• = node = fixed point that doesn't move



<https://zippy.gfycat.com/DirectMeekBustard.webm>

Models of the Atom

- Thomson – “Plum Pudding”
 - Why? Known that negative charges can be removed from atom.
 - Problem: Doesn't match scattering experiments
- Rutherford – Solar System
 - Why? Scattering showed a small, hard core.
 - Problem: electrons should spiral into nucleus in $\sim 10^{-11}$ sec.
- Bohr – quantized energy levels
 - Why? Explains spectral lines.
 - Problem: No reason for fixed energy levels
- deBroglie – electron standing waves
 - Why? Explains quantized energy levels
 - Problem: still only works for Hydrogen.
- Schrodinger – quantum wave functions
 - Why? Explains everything!
 - Problem: None (except that it's abstract)



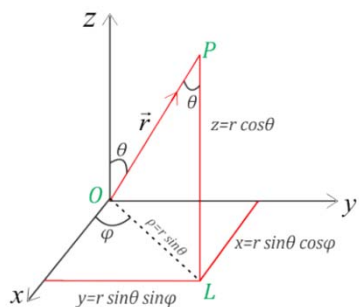
Outline for Day 13 - 14

Office hours: 2 - 3

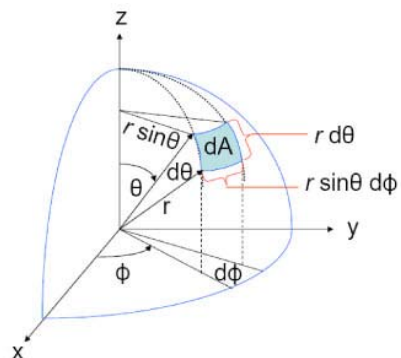
In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

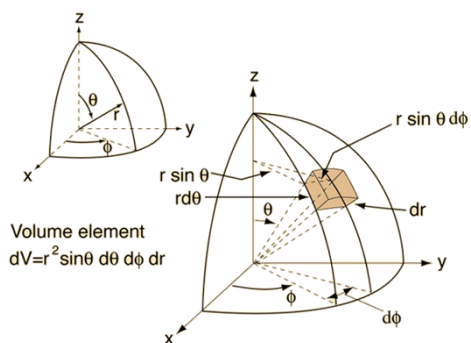
Spherical Coordinates



Spherical Coordinates



Spherical Coordinates



Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

Schrödinger equation in 3D + spherical coordinates

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) - \frac{\hbar^2}{2m} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + U\psi = E\psi \quad \text{Egu 8.49}^{\text{th}}$$

prove that: $\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r\psi$ ← equal

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r^2} \left[2r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial r^2} \right] = \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} \leftarrow \text{equal}$$

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} r\psi = \frac{1}{r} \frac{\partial}{\partial r} \left(\psi + r \frac{\partial \psi}{\partial r} \right) = \frac{1}{r} \left(\frac{\partial \psi}{\partial r} + \frac{\partial \psi}{\partial r} + r \frac{\partial^2 \psi}{\partial r^2} \right) = \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

Maybe a solution could look like

$$\psi = R(r)f(\theta)g(\phi)$$

$$\frac{\hbar^2}{2m} \frac{f(\theta)}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R(r)}{\partial r} \right) - \frac{\hbar^2}{2m} \left[\frac{R(r)}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f(\theta)}{\partial \theta} \right) + \frac{R(r)}{\sin^2 \theta} \frac{\partial^2 g(\phi)}{\partial \phi^2} \right] - \frac{kZe^2}{r} Rf g = E Rf g$$

multiply both sides by $-\frac{2mr^2 \sin^2 \theta}{\hbar^2 Rf g}$

if you can show that $f(x) = g(y)$ for all x and y
 $f(x) = g(y) = \text{constant}$

$$\frac{\sin^2 \theta}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) + \frac{\sin \theta}{f} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} + \frac{kZe^2}{r} \frac{2mr^2 \sin^2 \theta}{\hbar^2} = -\frac{2mr^2 E \sin^2 \theta}{\hbar^2}$$

$$\frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = -\frac{\sin^2 \theta}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) - \frac{\sin \theta}{f} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) - \frac{2mr^2 \sin^2 \theta}{\hbar^2} \left(E + \frac{kZe^2}{r} \right)$$

is a function of ϕ is a function of R and θ

$$\frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = \text{constant} = -m^2 \Rightarrow \frac{\partial^2 g}{\partial \phi^2} = -m^2 g \quad \text{let } g = e^{im\phi}$$

boundary condition for ϕ : $g(\phi + 2\pi) = g(\phi)$
 Single valued
 $e^{im\phi} = e^{im(\phi + 2\pi)} = e^{im\phi} e^{im2\pi} \quad m = 0, \pm 1, \pm 2, \dots$

Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

move all r dependence to one side

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{2mr^2}{\hbar^2} \left(E + \frac{kze^2}{r} \right) = \frac{-1}{f \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{df}{d\theta} \right) - \frac{1}{\sin^2 \theta} \frac{d^2 g}{d\phi^2}$$

depends only on r depends only on $\theta + \phi$

$= l(l+1)$

multiply both sides by $\frac{R}{r^2}$
 * use the identity for r derivative term

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \frac{l(l+1)R}{r^2} - \frac{2m}{\hbar^2} \frac{kze^2}{r} - \frac{2mr^2}{\hbar^2} \left(E + \frac{kze^2}{r} \right) \frac{R}{r^2}$$

$$\frac{d^2 R}{dr^2} = \frac{2m}{\hbar^2} \left[-\frac{kze^2}{r} + \frac{l(l+1)\hbar^2}{2mr^2} - E \right] R \quad \text{Egu 8.70}$$

Egu. 8.70 is a differential equation that has solutions tabulated in Table 8.2
 boundary conditions (not shown) require: $l < n$ l and n integers
 so $n=1 \Rightarrow l=0$
 $n=2 \Rightarrow l=0, 1$

that the energy is quantized

$$E_n = \frac{-m_e (ke^2)^2}{2\hbar^2} \frac{1}{n^2} = -13.6 \text{ eV} \frac{1}{n^2} \quad \text{Egu 8.74}$$

 $R_{nl}(r)$: The Radial Wave Functions for the atom

TABLE 8.2

The first few radial functions $R_{nl}(r)$ for the hydrogen atom. The variable ρ is an abbreviation for $\rho = r/a_0$ and a stands for a_0 .

	$n = 1$	$n = 2$	$n = 3$
$l = 0$	$\frac{2}{\sqrt{a^3}} e^{-\rho}$	$\frac{1}{\sqrt{2a^3}} \left(1 - \frac{1}{2}\rho \right) e^{-\rho/2}$	$\frac{2}{\sqrt{27a^3}} \left(1 - \frac{2}{3}\rho + \frac{2}{27}\rho^2 \right) e^{-\rho/3}$
$l = 1$		$\frac{1}{\sqrt{24a^3}} \rho e^{-\rho/2}$	$\frac{8}{27\sqrt{6a^3}} \left(1 - \frac{1}{6}\rho \right) \rho e^{-\rho/3}$
$l = 2$			$\frac{4}{81\sqrt{30a^3}} \rho^2 e^{-\rho/3}$

Look at $R_{10} = A e^{-r/a_0}$ $n=1, l=0$
 Eq 8.17

Let's prove this is a solution of the radial part of the Schrödinger equation with $l=0$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} R_{10} \right) = -\frac{2m}{\hbar^2} \left[\frac{\hbar^2}{2m} - \frac{2}{a_0} r \right] R_{10}$$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} A e^{-r/a_0} \right) = A \frac{\partial}{\partial r} \left[-\frac{r^2}{a_0} + r \left(\frac{1}{a_0} \right) e^{-r/a_0} \right]$$

$$= A \left[-\frac{2}{a_0} e^{-r/a_0} - \frac{1}{a_0} e^{-r/a_0} + r \left(\frac{1}{a_0} \right) e^{-r/a_0} \right]$$

$$= A \left[-\frac{3}{a_0} e^{-r/a_0} + \frac{r}{a_0} e^{-r/a_0} \right]$$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} R_{10} \right) = -\frac{2m}{\hbar^2} \left[\frac{\hbar^2}{2m} - \frac{2}{a_0} r \right] R_{10}$$

$$E_1 = -\frac{\hbar^2}{2m a_0^2} = -\frac{2^2 (m_e)^2 \hbar^2}{2m_e} = -\frac{2^2 (\hbar^2)^2 m_e}{2\hbar^2}$$

$$P(r) = |R_{nl}|^2 dV = |R_{nl}|^2 r^2 \sin\theta dr d\theta d\phi$$

$$P(x) = |\psi|^2 dx$$

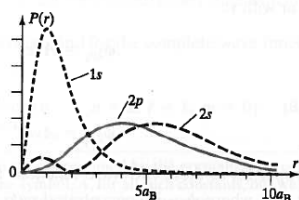
$$\frac{\hbar^2}{2m_e}$$

Radial Probability Density

$$P(r) = |R_{nl}|^2 r^2$$

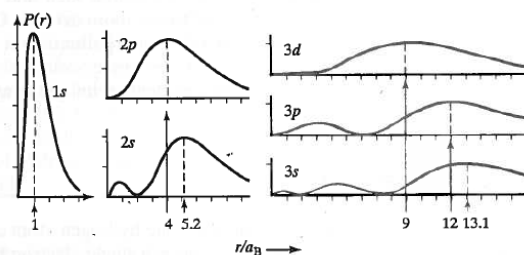
FIGURE 8.22

The radial probability density for the $2p$ states (solid curve). The most probable radius is $r = 4a_B$. For comparison, the dashed curves show the $1s$ and $2s$ distributions to the same scale.

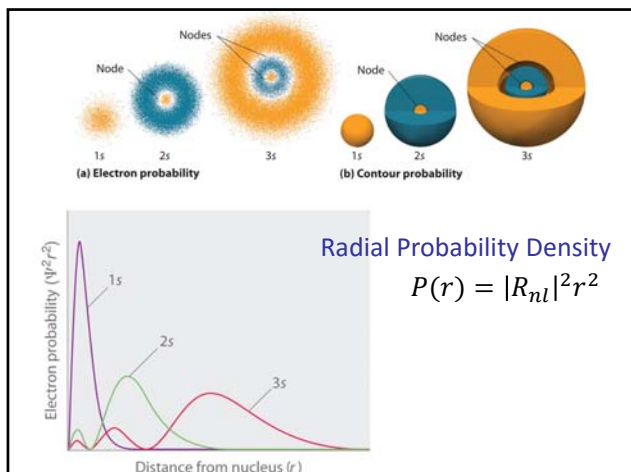


Radial Probability Density

$$P(r) = |R_{nl}|^2 r^2$$



most probable radius indicated for each state



Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

move all r dependence to one side

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{2mr^2}{\hbar^2} \left(E + \frac{kZe^2}{r} \right) = -\frac{1}{f \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) - \frac{1}{f \sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2}$$

depends only on r depends only on $\theta + \phi$

$$l(l+1) = -\frac{1}{f \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) - \frac{1}{f \sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2}$$

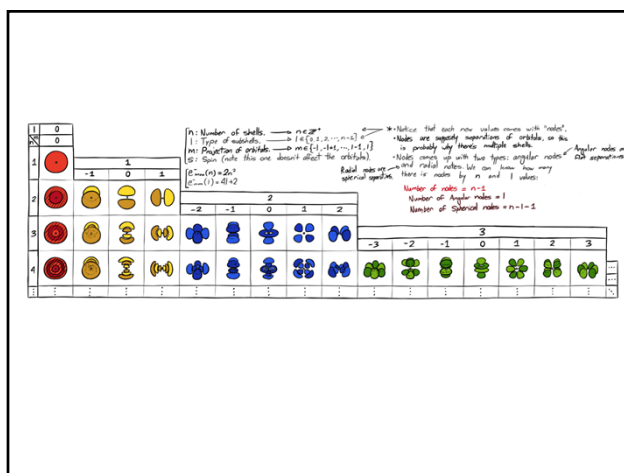
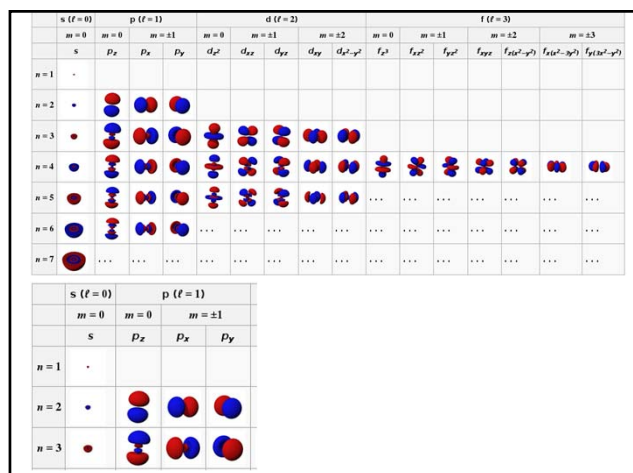
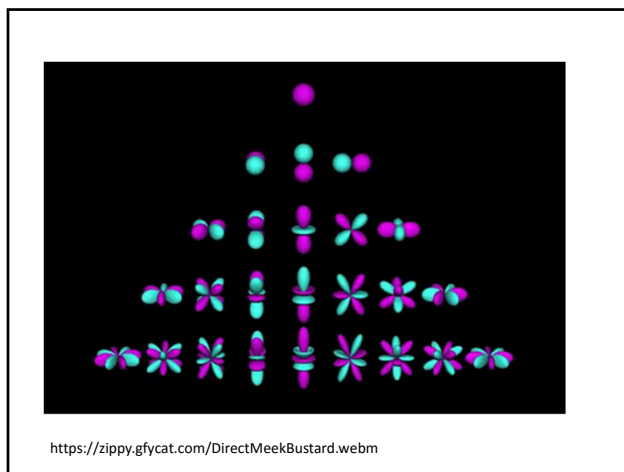
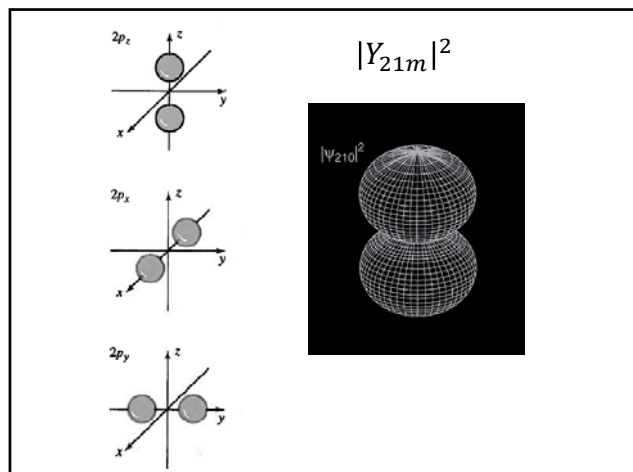
Solutions are in table 8.1
boundary conditions restrict values of m and l such that:
 $l \geq 0$ $|m| \leq l$ m, l integers
often angular functions are written together
 $Y_{lm}(\theta, \phi) = f_{lm}(\theta) g_m(\phi)$
 $= f_{lm}(\theta) e^{im\phi}$

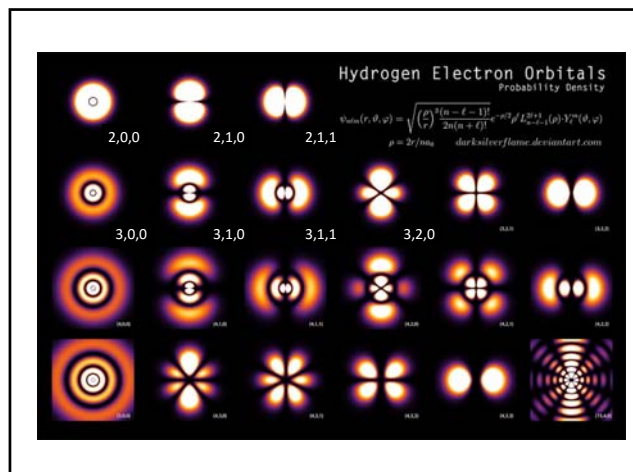
$Y_{lm}(\theta)$: The spherical harmonic wave functions

TABLE 8.1 $\Theta(\theta) = f(\theta)$ in our derivation in class
The first few angular functions $\Theta_{l,m}(\theta)$. The functions with m negative are given by $\Theta_{l,-m} = (-1)^m \Theta_{l,m}$.

	$l = 0$	$l = 1$	$l = 2$
$m = 0$	$\sqrt{1/4\pi}$	$\sqrt{3/4\pi} \cos \theta$	$\sqrt{5/16\pi} (3 \cos^2 \theta - 1)$
$m = 1$		$-\sqrt{3/8\pi} \sin \theta$	$-\sqrt{15/8\pi} \sin \theta \cos \theta$
$m = 2$			$\sqrt{15/32\pi} \sin^2 \theta$

$$Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta) e^{im\phi}$$





Outline for Day 13 - 14

Office hours: 2 - 3

In which we solve the Schrödinger Equation for the hydrogen atom.

- Bohr atom + de Broglie postulate
- Hydrogen atom – a 3 dimensional spherically symmetric object
 - Spherical coordinates
 - Schrodinger equation in 3 dimensions and spherical coordinates
- Solving the hydrogen atom
 - Solution for $g(\phi)$
 - Solution for $R(r)$ and allowed energy levels
 - Solution for $f(\theta)$
 - Complete solutions

Altogether the n

$$\psi_{nlm}(r, \theta, \phi) = \underbrace{C_{nlm}}_{\text{normalization constant}} \underbrace{R_{nl}(r)}_{\text{table 8.2}} \underbrace{f_l(\theta)}_{\text{table 8.1}} \underbrace{e^{im\phi}}_{\text{constant}}$$

$n = 1, 2, 3, \dots$
 $l \geq 0 \quad l < n \quad l = 0, 1, 2, 3, \dots, (n-1)$
 $|m| \leq l \quad m = -l, -(l+1), -(l+2) \dots 0, 1, 2, \dots, +l$

s p d f

$R_{nl}(r)$: The Radial Wave Functions for the atom

TABLE 8.2

The first few radial functions $R_{nl}(r)$ for the hydrogen atom. The variable ρ is an abbreviation for $\rho = r/a_0$ and a stands for a_0 .

	$n = 1$	$n = 2$	$n = 3$
$l = 0$	$\frac{2}{\sqrt{a^3}} e^{-\rho}$	$\frac{1}{\sqrt{2a^3}} \left(1 - \frac{1}{2}\rho\right) e^{-\rho/2}$	$\frac{2}{\sqrt{27a^3}} \left(1 - \frac{2}{3}\rho + \frac{2}{27}\rho^2\right) e^{-\rho/3}$
$l = 1$		$\frac{1}{\sqrt{24a^3}} \rho e^{-\rho/2}$	$\frac{8}{27\sqrt{6a^3}} \left(1 - \frac{1}{6}\rho\right) \rho e^{-\rho/3}$
$l = 2$			$\frac{4}{81\sqrt{30a^3}} \rho^2 e^{-\rho/3}$

$Y_{lm}(\theta)$: The spherical harmonic wave functions

TABLE 8.1

$\Theta(\theta) = f(\theta)$ in our derivation in class

The first few angular functions $\Theta_{l,m}(\theta)$. The functions with m negative are given by $\Theta_{l,-m} = (-1)^m \Theta_{l,m}$.

	$l = 0$	$l = 1$	$l = 2$
$m = 0$	$\sqrt{1/4\pi}$	$\sqrt{3/4\pi} \cos \theta$	$\sqrt{5/16\pi} (3 \cos^2 \theta - 1)$
$m = 1$		$-\sqrt{3/8\pi} \sin \theta$	$-\sqrt{15/8\pi} \sin \theta \cos \theta$
$m = 2$			$\sqrt{15/32\pi} \sin^2 \theta$

$$Y_{lm}(\theta, \phi) = \Theta_{lm}(\theta) e^{im\phi}$$

Quantum number l :	0	1	2	3
Magnitude L :	0	$\sqrt{2}\hbar$	$\sqrt{6}\hbar$	$\sqrt{12}\hbar$
Code letter:	s	p	d	f

$E = 0$	$\frac{4s}{16}$ (1)	$\frac{4p}{32}$ (3)	$\frac{4d}{32}$ (5)	$\frac{4f}{72}$ (7)
$E_4 = -E_R/16$	$\frac{4s}{16}$ (1)	$\frac{4p}{32}$ (3)	$\frac{4d}{32}$ (5)	
$E_3 = -E_R/9$		$\frac{4p}{32}$ (3)	$\frac{4d}{32}$ (5)	
$E_2 = -E_R/4$		$\frac{4p}{32}$ (3)	$\frac{4d}{32}$ (5)	
$E_1 = -E_R$	$\frac{4s}{16}$ (1)			

FIGURE 8.16
Energy-level diagram for the hydrogen atom, with energy plotted upward and angular momentum to the right. The letters s, p, d, f, ... are code letters traditionally used to indicate $l = 0, 1, 2, 3, \dots$. (Energy spacing not to scale.)

code letter: s p d f g h i
quantum number l : 0 1 2 3 4 5 6

$$n = 1, 2, 3, \dots$$

$$l < n$$

$$|m| \leq l$$

Let's look at our complete solution:

$$\psi_{210}(r, \theta, \phi) = C_{210} R_{21}(r) Y_{21}(\theta, \phi)$$

$$= C_{210} \frac{1}{\sqrt{24} a_0^3} \left(\frac{r}{a_0} \right)^2 \cos \theta e^{-r/2a_0} e^{im\phi}$$

find C_{210}

$$1 = \int |\psi|^2 dV = \int_0^\infty \int_0^\pi \int_0^{2\pi} C_{210}^2 \frac{1}{24 a_0^3} \left(\frac{r}{a_0} \right)^4 \cos^2 \theta e^{-r/a_0} r^2 \sin \theta dr d\theta d\phi$$

$$1 = C_{210}^2 \frac{1}{24 a_0^3} \int_0^\infty r^4 e^{-r/a_0} dr \int_0^\pi \cos^2 \theta \sin \theta d\theta \int_0^{2\pi} d\phi$$

Well known integrals

$$\int_0^\infty r^4 e^{-r/a_0} dr = 24 a_0^5$$

$$\int_0^\pi \cos^2 \theta \sin \theta d\theta = \frac{4}{3}$$

$$\int_0^{2\pi} d\phi = 2\pi$$

$$1 = C_{210}^2 \frac{1}{24 a_0^3} (24 a_0^5) \left(\frac{4}{3} \right) (2\pi)$$

$$C_{210}^2 = \frac{1}{16\pi a_0^2}$$

$$C_{210} = \frac{1}{4\sqrt{2}\pi a_0^2}$$